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CONSTRAINED OPTIMAL CONTROLLER FOR LINEAR SYSTEMS WITH STATE- AND CONTROL-DEPENDENT DISTURBANCE

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16. Abstract

The design of a fixed, constrained, optimal linear controller for a linear system with an unknown state- and control-dependent disturbance is considered. The problem is posed with the additional constraints that the dynamic controller uses only noise-corrupted outputs, and that its dimension is significantly lower than that of a Kalman filter. The unknown disturbance is viewed as an adversary which tries to maximize a performance criterion: a criterion that the controller gains attempt to minimize. The optimal controller gains are determined by solving a nonlinear matrix two-point boundary value problem.

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INTRODUCTION

This study considers the mathematical problem of designing a constrained optimal, dynamic controller for linear systems with parameter uncertainty. Usually exact values of the system parameters are not known to the designer, and any modeling error is usually taken into account as additive white noise disturbance in the input channel, assuming that the correlation time of the noise process is much smaller than that of the controlled system. In the case of wide deviations of parameters, the intensities of the disturbances vary with the deviations of state and control matrices. Thus the previous assumption, that the correlation time of the noise process is negligible compared to that of the controlled system, is no longer valid.

McLane (1) has shown that control-dependent noise occurs in modeling the thrust misalignment in a gas-jet thrusting system for the attitude control of a satellite. The state-dependent noise occurs in the momentum exchange method for regulating the angular precession of rotating spacecraft. Krasovskii (2), Wonham (3), McLane (4), and others [(1), (5), (6), and (7)] have proposed a stochastic formulation of the above problem, assuming that the intensities of the state- and control-dependent noise are known and have determined the time-varying feedback gain by minimizing a quadratic criterion.

In this study, an alternative approach to the above problem is described. This approach assumes that nominal values of system matrices are available, although their precise values or variations are unknown, and it prescribes a minimax design for a time-invariant dynamic controller. The technique presented in this paper considers the uncertain parameters and their deviations to be an adversary which seeks to maximize a performance criterion, a criterion which the controller attempts to minimize. This minimaximization procedure leads to a conservative design, although it is argued that the performance criterion reflects a meaningful physical situation. Note that the effect of uncertain parameters, the state- and control-dependent noise, has a destabilizing influence on the control system; that is, it tends to maximize the performance criterion (4).

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It is to be noted here that the system need not be governed by Îto stochastic differential equations since uncertain parameters or parameter variations from nominal are considered in this paper as essentially nonrandom (8). In addition, the problem is posed with two additional constraints; first, that only the noise-corrupted outputs, not all the states, are available; and second, that the feedback gains (controller gains) are time-invariant. The second constraint seems more practical from the viewpoint of implementing the controller. Since it has been observed in references (3) and (5) that linearly state- and control-dependent noise of sufficient intensities may make stabilization by linear full-state feedback impossible (let alone the output feedback), even if the nominal system matrix pair (A_0, B_0) is controllable in the usual sense, it is believed that a dynamic controller which uses noise-corrupted outputs has a better chance of stabilizing the system.

Although the estimation of state from limited measurements may be provided by a Kalman filter, the problem becomes difficult in this case, due to the presence of parameter uncertainty. In addition, the dimension of the filter may be unnecessarily large. This subject has been the motivation of a number of papers in deterministic optimal design [(9), (10), and (11)]. Minimax controller designs have been proposed by many authors [(12), (13), and (14)]. However, most of these papers allow the control or its opponent to employ full-state feedback.

The problem is treated here by minimizing, with respect to feedback matrices, and maximizing, with respect to parameter uncertainty, a quadratic performance index which involves state, control, and parameter deviations. The necessary conditions which result are a set of nonlinear two point boundary value problems (TPBVP). The present formulation also permits the inclusion of situations where the passband of disturbances is not necessarily large compared to that of the controlled system. This approach is a generalization of the basic concepts advanced by Wonham (3), McLane (4), and others, in that the dynamic controller, rather than the output feedback, has a better chance of stabilizing the system and the formulation permits inclusion of colored state- and control-dependent noise.

SYSTEM DESCRIPTION AND PROBLEM FORMULATION

The system to be controlled is a linear system with state vector $\underline{x}(t)$, output vector $\underline{y}(t)$, control vector $\underline{u}(t)$, and compensator state vector $\underline{z}(t)$ related by

$$\dot{\underline{x}} = A_0\underline{x} + B_0\underline{u} + (A - A_0)\underline{x} + (B - B_0)\underline{u} + D\underline{w}(t), \quad \text{Plant state equation} \tag{1}$$

$$\dot{\underline{z}} = F\underline{z} + G\underline{y}, \quad \text{Compensator equation} \tag{2}$$

$$\underline{y} = C\underline{x} + \underline{v}(t), \quad \text{Measurement equation} \tag{3}$$

$$\underline{u} = H\underline{z} + N\underline{y}, \quad \text{Control equation.} \tag{4}$$

A_0 and B_0 are nominal representations of matrices A and B, respectively, which may contain certain elements. Vectors $\underline{x}$, $\underline{z}$, $\underline{y}$, $\underline{u}$, and $\underline{w}$ have dimension n, p, q, r, and d, respectively. All matrices are time-invariant and have compatible dimensions; $\underline{v}(t)$ and $\underline{w}(t)$ are sample functions of zero mean, stationary, white-noise processes satisfying

$$E\,[\underline{w}(t_1)\underline{w}^T(t_2)] = W\delta(t_1 - t_2),\ E\,[\underline{v}(t_1)\underline{v}^T(t_2)] = V\delta(t_1 - t_2),$$
$$E\,[\underline{w}(t_1)\underline{v}^T(t_2)] = U^T\delta(t_1 - t_2). \tag{5}$$

Furthermore, the initial state of the plant is assumed to be random with a known mean and convariance. Having specified the dimension of the compensator, the designer is free to choose the matrices F, G, H, and N and the initial state $\underline{z}_0$.

The system equations (1) through (4) may be rewritten so that F, G, H, and N appear as elements of a single matrix. For this purpose the following real vectors and matrices are defined with the dimensions of partitions indicated:

$$\underline{m} = \begin{array}{cc} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},\ E\,[\underline{m}(0)] = \underline{m}_0,\ \underline{s} = \begin{array}{cc} \begin{bmatrix} \underline{v} \\ \underline{w} \end{bmatrix} & \begin{array}{c} q \\ d \end{array} \end{array},\ \hat{A}_0 = \begin{array}{cc} \begin{array}{cc} n & p \end{array} & \\ \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},$$

$$\hat{B}_0 = \begin{array}{cc} \begin{array}{cc} r & p \end{array} & \\ \begin{bmatrix} B_0 & 0 \\ 0 & I_p \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},\ Q = \begin{array}{cc} \begin{array}{cc} n & p \end{array} & \\ \begin{bmatrix} Q_x & Q_{xz} \\ Q_{zx} & Q_z \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},\ \hat{C} = \begin{array}{cc} \begin{array}{cc} n & p \end{array} & \\ \begin{bmatrix} C & 0 \\ 0 & I_p \end{bmatrix} & \begin{array}{c} q \\ p \end{array} \end{array},\ \hat{D} = \begin{array}{cc} \begin{array}{cc} q & d \end{array} & \\ \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},$$

$$\hat{N} = \begin{array}{cc} \begin{array}{cc} q & p \end{array} & \\ \begin{bmatrix} N & H \\ G & F \end{bmatrix} & \begin{array}{c} r \\ p \end{array} \end{array},\ \hat{R} = \begin{array}{cc} \begin{array}{cc} r & p \end{array} & \\ \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} r \\ p \end{array} \end{array},\ \hat{I} = \begin{array}{cc} \begin{array}{cc} q & d \end{array} & \\ \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} q \\ p \end{array} \end{array},\ \hat{\nu}^T = \begin{array}{cc} \begin{array}{cc} r & p \end{array} & \\ \begin{bmatrix} \nu^T & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} q \\ p \end{array} \end{array}$$

$$E\,\left[s(t_1)s^T(t_2)\right] = S\delta(t_1 - t_2),\ S = \begin{bmatrix} V & U \\ U^T & W \end{bmatrix},\ M_0 = E\left[\underline{m}(0)\underline{m}(0)^T\right],$$

$$(A - A_0) = W_1,\ (B - B_0) = W_2,\ \hat{W}_1 = \begin{array}{cc} \begin{array}{cc} n & p \end{array} & \\ \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} n \\ p \end{array} \end{array},\ \hat{W}_2 = \begin{array}{cc} \begin{array}{cc} r & p \end{array} & \\ \begin{bmatrix} W_2 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} r \\ p \end{array} \end{array}, \tag{6}$$

where I_p and I_q are identity matrices of dimension p and q respectively. Equations (1) through (4) and (6) combine to give

$$\dot{\underline{m}} = (\hat{A}_0 + \hat{B}_0\hat{N}\hat{C})\underline{m} + (\hat{D} + \hat{B}_0\hat{N}\hat{I})s + (\hat{W}_1 + \hat{W}_2\hat{N}\hat{C})\underline{m} + \hat{W}_2\hat{N}\hat{I}s. \tag{7}$$

The basic problem now reduces to select the feedback matrix $\hat{N}$, in order to provide acceptable performance despite the presence of parameter uncertainties $\hat{W}_1$ and $\hat{W}_2$. In order to determine $\hat{N}$, a quadratic performance criterion involving state vector $\underline{m}$, control vector $\underline{u}$, and parameter uncertainty $\hat{W}_1$, $\hat{W}_2$ is defined. This criterion is then maximized with respect to $\hat{N}$. The uncertain parameters are therefore permitted to assume their worst values, limited only by penalties included in the criterion. Although this type of minimax design is often conservative, it has the advantage of requiring relatively meager information about uncertain parameters. It should also be noted that the colored state- and control-dependent noise can be modeled by using additional states in the plant and suitably modifying matrix C. The dimension of $\underline{z}$ need not be modified, however, and even in the full state estimation problem it may be retained as the original $\underline{x}$ dimension.

It should be noted that $\hat{W}_1$ and $\hat{W}_2$ will normally contain many zero entries. In order to restrict variation of $\hat{W}_1$ and $\hat{W}_2$ to the nonzero terms, it is helpful to decompose $\hat{W}_1$ and $\hat{W}_2$ in a simple way. If a matrix W has n rows and m columns, it is always possible to express W as

$$W = \sum_{i=1}^{\ell} D_i G_i C_i , \tag{8}$$

where matrices G_i contain the nonzero terms and only the nonzero terms of W, D_i, and C_i are known matrices, ℓ is the minimum numbers of rows and columns, in any combination, which are required to cover the nonzero terms of W. Clearly, $\ell \leqslant \min(n, m)$. Two examples illustrate this expansion.

Example 1. *nth* order system with *1st* order compensator:

$$\hat{W} \triangleq \left[\begin{array}{c|c} W & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & 0 & 0 \\ W_{1n} & W_{2n} & & W_{nn} & 0 \\ \hline 0 & 0 & & & 0 \end{array}\right] = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} W_{1n}, W_{2n}, \ldots W_n \end{bmatrix} \left[\begin{array}{c|c} I & 0 \end{array}\right].$$

$$= D_1 G_1 C_1$$

$\hat{W}_1$ may assume this form whenever matrix A is in companion form and a first order compensator is used.

A single term of equation (8) is adequate here since W has but one nonzero row.

Example 2.

$$\left[\begin{array}{ccc|c} W_{11} & 0 & 0 & 0 \\ W_{21} & 0 & 0 & 0 \\ W_{31} & W_{32} & W_{33} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_{11} \\ W_{21} \\ W_{31} \end{bmatrix} [1\ 0\ 0\ 0]$$

$$+\begin{bmatrix}0\\0\\1\\0\end{bmatrix}[W_{32}\ W_{33}]\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} = D_1 G_1 C_1 + D_2 G_2 C_2 .$$

In this case, nonzero elements of W are covered by one column and one row, and hence the expansion has two terms. Care has been taken to insure that W_{31} is included only once. By proper scaling with C_i for a row term or D_i for a column term of equation (8), the range of parameter variations expected for all elements of G_i can be made the same. This is useful in properly penalizing parameter variations in the performance criterion. It should be noted that the construction suggested assures that C_i has maximum rank.

It is assumed, for simplicity in the developments which follow, that

$$\hat{W}_1 = D_1 G_1 C_1 , \ \hat{W}_2 = D_2 G_2 C_2 , \tag{9}$$

that is, W_1 and W_2 are assumed to contain a single nonzero row or column. However, the procedure employed to obtain necessary conditions for a minimax solution can also be used in a straightforward manner when W_1 and W_2 require the more general expansion of equation (8). If equation (9) is substituted in equation (7), the result is a state equation in terms of feedback matrix $\hat{N}$ and "uncertainty" matrices G_1 and G_2,

$$\underline{\dot{m}} = (\hat{A}_0 + \hat{B}_0 \hat{N} \hat{C})\underline{m} + (\hat{D} + \hat{B}_0 \hat{N} \hat{I})\underline{s} + (D_1 G_1 C_1 + D_2 G_2 C_2 \hat{N} \hat{C})\underline{m} + D_2 G_2 C_2 \hat{N} \hat{I} \underline{s}$$

$$= (\hat{A}_0 + \hat{B}_0 \hat{N} \hat{C} + D_1 G_1 C_1 + D_2 G_2 C_2 \hat{N} \hat{C})\underline{m} + (\hat{D} + \hat{B}_0 \hat{N} \hat{I} + D_2 G_2 C_2 \hat{N} \hat{I})\underline{s} . \tag{10}$$

Matrix $\hat{N}$ will be determined using the performance criterion

$$J(\hat{N}, G_1, G_2, \underline{z}_0) = \frac{1}{2} E \int_{t_0}^{t_f} \begin{bmatrix}\underline{x}\\ \underline{z}\end{bmatrix}^T \begin{bmatrix} Q_x & Q_{xz}\\ Q_{zx} & Q_z\end{bmatrix}\begin{bmatrix}\underline{x}\\ \underline{z}\end{bmatrix} + \underline{u}^T R \underline{u}\, dt - \frac{1}{2} \mathrm{Tr}\left[K_1 G_1 G_1^T + K_2 G_2 G_2^T\right]$$

$$= \frac{1}{2} E \int_{t_0}^{t_f} \left\{\underline{m}^T \left[Q + \hat{C}^T \hat{N}^T \hat{R} \hat{N} \hat{C}\right] \underline{m} + 2\underline{m}^T \begin{bmatrix} C^T N^T R N\\ H^T R N\end{bmatrix} \underline{v}\right.$$

$$\left. + \frac{1}{2}\underline{v}^T N^T R N \underline{v}\right\} dt - \frac{1}{2}\mathrm{Tr}\left[K_1 G_1 G_1^T + K_2 G_2 G_2^T\right] . \tag{11}$$

R, K_1 and K_2 are positive definite, symmetric matrices. Q is symmetric and positive semidefinite; R and Q are constant. Tr in equation (11) denotes the trace. The trace terms have been introduced to limit the variation of the uncertain parameters G_1 and G_2. K_1 and K_2 may provide some flexibility in weighing elements of G_1 and G_2, but this is best accomplished by proper sealing with D_1, C_1, D_2, and C_2 as previously suggested. Frequently, K_1 and K_2 will merely be scalars. The minus sign is necessary in equation (11) because the criterion is to be maximized with respect to G_1 and G_2. Initial time $t_0 = 0$ and final time $t_f = \infty$ are of primary interest. It must be assumed, therefore, that the dimension p of the compensator is adequate to stabilize the system so that the integral component of equation (11) remains finite as $t_f \to \infty$.

It should be noted

$$\frac{1}{2} E\left[\underline{v}^T N^T R N \underline{v}\right] = \mathrm{Tr}\left[N^T R N E(\underline{v}\ \underline{v}^T)\right] = \frac{1}{2} \mathrm{Tr}\left[N^T R N V\right] \delta(0),$$

$\delta(0)$ is the impulse fulsion. This makes an unbounded contribution to J unless $\mathrm{Tr}[N^T RNV] = 0$. We shall assume that R and V are either positive definite or zero. Hence it is clear that either

$$\text{(a)}\quad R = 0, \quad \text{(b)}\ V = 0,\ \underline{v}(t) = 0, \quad \text{or (c)}\quad N = 0$$

is required for a finite L. This result is not surprising. When the output is corrupted with white noise, design via quadratic cost involving $u^T Ru$ will lead to zero output feedback gain. What it really means is that the noise must be filtered before it is fed back to the system. Of course, nonzero output feedback gain can be obtained by setting $R = 0$, meaning J is independent of U or the noise-free measurement. In any case described above, equation (11) may be replaced by

$$J(\hat{N}, G_1, G_2, \underline{z}(t_0)) = \frac{1}{2} E\int_{t_0}^{t_f} \left\{\underline{m}^T \left[Q + \hat{C}^T \hat{N}^T \hat{R} \hat{N} \hat{C}\right] \underline{m} + \mathrm{Tr}\left[\hat{\nu}^T \hat{N}\right]\right\} dt - \frac{1}{2} \mathrm{Tr}\left[K_1 G_1 G_1^T + K_2 G_2 G_2^T\right] \tag{12}$$

Addition of $\mathrm{Tr}\,[\hat{\nu}^T \hat{N}]$ merely appends the constraint $N = 0$ when $R \neq 0$ and $V \neq 0$. Otherwise $\hat{\nu} = 0$, and $N = 0$.

MINIMAXIMIZATION OF THE FINITE INTERVAL CRITERION, J

Conditions to be satisfied by the matrix $\hat{N}$, G_1, G_2 and initial state $\underline{z}_0$ which minimaximize J will be obtained using an approach presented in references (15) and (16).

Let

$$J = E[\hat{J}], \quad \hat{J} = \int_0^{t_f} L\, dt. \tag{13}$$

If $\hat{N}$, G_1, G_2 and $\underline{z}_0$ are to be extremum, then

$$\frac{\partial J}{\partial \hat{N}} = E\left[\frac{\partial \hat{J}}{\partial \hat{N}}\right] = 0,\ \frac{\partial J}{\partial G_1} = E\left[\frac{\partial \hat{J}}{\partial G_1}\right] = 0,\ \frac{\partial J}{\partial G_2} = E\left[\frac{\partial \hat{J}}{\partial G_2}\right] = 0,\ \frac{\partial J}{\partial \underline{z}_0} = E\left[\frac{\partial \hat{J}}{\partial \underline{z}_0}\right] = 0, \quad (14)$$

where the interchange of expectation and differentiation is assumed to be valid (17). The partial derivatives in equation (14) may be evaluated using the following well-known results (18).

Lemma:

If
$$J = J(\underline{x}(t_0)) = \int_{t_0}^{t_f} L(\underline{x}, t)\, dt + P(\underline{x}, t_f),$$

where
$$\dot{\underline{x}} = \underline{f}(\underline{x}, t),$$

then
$$\frac{\partial J}{\partial \underline{x}(t_0)} = \underline{\lambda}(t_0),$$

where
$$\dot{\underline{\lambda}} = \frac{\partial H}{\partial \underline{x}} = -\frac{\partial}{\partial \underline{x}}(L + \underline{\lambda}^T f),\ \underline{\lambda}(t_f) = \frac{\partial P(\underline{x}(t_f), t_f)}{\partial x}$$

This lemma follows from the variational calculus where the first variation of J with respect to $\underline{x}(t_0)$ is $\lambda^T(t_0)\, \delta\underline{x}(t_0)$.

In order to apply the lemma here, elements of N, G_1 and G_2 are treated as additional states which satisfy

$$\dot{\hat{N}} = 0,\ \dot{G}_1 = 0,\ \dot{G}_2 = 0. \quad (15)$$

Vector multiplier $\underline{\lambda}_m$ will be used for the regular state constraint of equation (10), and matrix multiplier $\hat{\Lambda}_N(t)$, $\Lambda_{G_1}(t)$, and $\Lambda_{G_2}(t)$ will be used for constraints of equation (15). Note that the Hamiltonian H in the lemma will be independent of $\hat{\Lambda}_N$, Λ_{G_1}, Λ_{G_2} due to equation (15). Thus the Hamiltonian is

$$H = \frac{1}{2}\underline{m}^T\left[Q + \hat{C}^T\hat{N}^T\hat{R}\hat{N}\hat{C}\right]\underline{m} + \underline{\lambda}_m^T\left[\left(\hat{A}_0 + \hat{B}_0\hat{N}\hat{C}\right)\underline{m} + \left(\hat{D} + \hat{B}_0\hat{N}\hat{I}\right)s\right.$$
$$\left. + \left(D_1 G_1 C_1 + D_2 G_2 C_2 \hat{N}\hat{C}\right)\underline{m} + D_2 G_2 C_2 \hat{N}\hat{I}\underline{s}\right] + \mathrm{Tr}\left[\hat{\nu}^T\hat{N}\right]$$
$$= \mathrm{Tr}\left[\frac{1}{2}\overset{*}{Q}\underline{m}\,\underline{m}^T\right] + \mathrm{Tr}\left[\overset{*}{A}_\delta\underline{m}\,\underline{\lambda}_m^T + \left(\hat{D} + \hat{B}_0\hat{N}\hat{I} + D_2 G_2 C_2\hat{N}\hat{I}\right)\underline{s}\ \lambda_m^T\right] \quad (16)$$
$$+ \mathrm{Tr}\left[\hat{\nu}^T\hat{N}\right]$$

where

$$\dot{\underline{\lambda}}_m = -\frac{\partial H}{\partial \underline{m}} = -\left(\hat{A}_0 + \hat{B}_0 \hat{N} \hat{C} + D_1 G_1 C_1 + D_2 G_2 C_2 \hat{N} \hat{C}\right)^T \underline{\lambda}_m - \overset{*}{Q} \underline{m},$$

$$= -\overset{*}{A}{}_0^T \underline{\lambda}_m - \overset{*}{Q} \underline{m}, \; \underline{\lambda}_m (t_f) = 0. \tag{17}$$

$$\dot{\hat{\Lambda}}_N = -\frac{\partial H}{\partial \hat{N}} = -\hat{R} \hat{N} \hat{C} \underline{m} \underline{m}^T \hat{C}^T - \left(\hat{B}_0 + D_2 G_2 C_2\right)^T \underline{\lambda}_m \, m^T \hat{C}^T - \hat{\nu}$$

$$- \hat{B}_0^T \underline{\lambda}_m \underline{s}^T \hat{I}, \; \hat{\Lambda}_N (t_f) = 0 \tag{18}$$

$$\dot{\Lambda}_{G_1} = -\frac{\partial H}{\partial \Lambda_{G_1}} = -D_2^T \underline{\lambda}_m \underline{m}^T C_1^T, \; \Lambda_{G_1} (t_f) = -(K_1 G_1), \tag{19}$$

$$\dot{\Lambda}_{G_2} = -\frac{\partial H}{\partial \Lambda_{G_2}} = -D_2^T \underline{\lambda}_m \left[\underline{m}^T \hat{C}^T + \underline{s}^T \hat{I}\right] \hat{N}^T C_2^T, \; \Lambda_{G_2} (t_f) = -K_2 G_2, \tag{20}$$

where

$$\overset{*}{A}_0 = \left(\hat{A}_0 + \hat{B}_0 N \hat{C} + D_1 G_1 C_1 + D_2 G_2 C_2 \hat{N} \hat{C}\right), \tag{21}$$

$$\overset{*}{Q} = \left(Q + \hat{C}^T \hat{N}^T \hat{R} \hat{N} \hat{C}\right) \tag{22}$$

The gradient matrix expressions in equations (18), (19), and (20) have been obtained using the rules in reference (19).

The lemma, the necessary conditions of equation (14), and integrated forms of equations (18), (19), and (20) combine to give

$$E\left[\frac{\partial J}{\partial \hat{N}}\right] = E\left[\hat{\Lambda}_N (t_0)\right] = E \int_{t_0}^{t_f} \left[\hat{R} \hat{N} \hat{C} \underline{m} \underline{m}^T \hat{C}^T \right.$$

$$\left. + \left(B_0 + D_2 G_2 C_2\right)^T \underline{\lambda}_m \underline{m}^T C^T \hat{B}_0^T \underline{\lambda}_m s^T \hat{I} + \hat{\nu}\right] dt = 0, \tag{23}$$

$$E\left[\frac{\partial J}{\partial G_1}\right] = E\left[\Lambda_{G_1} (t_0)\right] = E \int_{t_0}^{t_f} \left[D_1^T \underline{\lambda}_m \underline{m}^T C_1^T\right] dt - K_1 G_1 = 0, \tag{24}$$

$$E\left[\frac{\partial J}{\partial G_2}\right] = E\left[\Lambda_{G_2}(t_0)\right] = -E\int_{t_0}^{t_f} D_2^T \underline{\lambda}_m \left(m^T \hat{C}^T + s^T \hat{I}^T\right) \hat{N}^T C_2^T\, dt - K_2 G_2 = 0 . \quad (25)$$

Equations (23), (24), and (25) can be rearranged to yield simultaneous expressions for N, G_1, and G_2 which involve integrals of E $[\underline{m}\, \underline{m}^T]$ and E $[\underline{\lambda}_m\, \underline{m}^T]$. In order to evaluate these integrals, assume that

$$\underline{\lambda}_m(t) = K(t)\underline{m}(t) + \underline{\eta}(t) , \quad (26)$$

satisfies equation (17) for some choice of K(t) and $\underline{\eta}(t)$. Substitutions of equations (26) and (10) in equation (17) indicate that the assumption is valid provided

$$\dot{K} + K\overset{*}{A}_0 + \overset{*}{A}{}_0^T K + \overset{*}{Q} = 0, \quad K(t_f) = 0, \quad (27)$$

and

$$\dot{\underline{\eta}} + \overset{*}{A}{}_0^T \underline{\eta} + K(\hat{D} + \hat{B}_0 \hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I})\underline{s} = 0, \ \eta(t_f) = 0. \quad (28)$$

By expressing the solutions of equations (10) and (28) using the transition matrix associated with A_0, it is straightforward to show

$$E\left[\underline{m}(t)\underline{\eta}^T(t)\right] = 0,$$

$$E\left[\underline{\eta}(t)\underline{s}^T(t)\right] = K(t)\ E\left[\underline{m}(t)\underline{s}^T(t)\right] = \frac{1}{2}K(t)\left(\hat{D} + \hat{B}_0 \hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}\right) S ,$$

$$E\left[\underline{\lambda}_m(t)\underline{m}^T(t)\right] = K(t)P(t),$$

$$E\left[\underline{\lambda}_m(t)s^T(t)\right] = K(t)\left(\hat{D} + \hat{B}_0 \hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}\right) S, \quad (29)$$

where

$$P(t) \triangleq E\left[\underline{m}(t)\underline{m}^T(t)\right] \quad (30)$$

is governed by

$$\dot{P} = \overset{*}{A}_0 P + P\overset{*}{A}{}_0^T + \left(\hat{D} + \hat{B}_0 \hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}\right) S \left(\hat{D} + \hat{B}_0 \hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}\right)^T , \quad (31)$$

and

$$P(0) = E\left[\underline{m}(0)\underline{m}^T(0)\right] = M_0 . \quad (32)$$

Hence, equations (23), (24), and (25) become

$$\hat{R}\hat{N}\hat{C}\left[\int_0^{t_f} P(t)\,dt\right]\hat{C}^T + \left(B_0 + D_2 G_2 C_2\right)^T\left[\int_0^{t_f} K(t)P(t)\,dt\right]\hat{C}^T$$

$$+ \hat{B}_0^T\left[\int_0^{t_f} K(t)\,dt\right]\left(\hat{D} + \hat{B}_0\hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}\right) S\hat{I} + \int_0^{t_f} \hat{\nu}\,dt = 0\,, \quad (33)$$

$$G_1 = K_1^{-1} D_1^T\left[\int_0^{t_f} K(t)P(t)\,dt\right] C_1^T\,, \quad (34)$$

$$G_2 = -K_2^{-1} D_2^T\left[\int_0^{t_f} K(t)\left\{P(t)\hat{C}^T + \hat{D} + \hat{B}_0\hat{N}\hat{I} + D_2 G_2 C_2 \hat{N}\hat{I}S\hat{I}^T\right\}\hat{N}^T C_2^T\,dt\right]. \quad (35)$$

When $\underline{z}_0$ is assigned arbitrarily, $\hat{N}$ is determined by simultaneous solution of equations (27) and (31) through (35). As $t_f \to \infty$ in equations (33), (34), and (35), the integrals are dominated by steady-state values of K and P. As $t_f \to \infty$, however, J will not in general remain finite.

If $\underline{z}_0$ is to be chosen optimally, an expression for $E[\underline{\lambda}_m(0)]$ will be required. From equations (26), (28), and (6) it is clear that

$$E\left[\underline{\lambda}_m(0)\right] = E\begin{bmatrix}\underline{\lambda}_x(0)\\ \underline{\lambda}_z(0)\end{bmatrix} = K(0)\,E\begin{bmatrix}\underline{x}(0)\\ \underline{z}(0)\end{bmatrix} + E\left[\eta(0)\right] = K(0)\begin{bmatrix}\underline{x}_0\\ \underline{z}_0\end{bmatrix}. \quad (36)$$

If matrices K and P are partitioned according to

$$K = K^T = \begin{array}{c} \begin{array}{cc} n & p \end{array} \\ \begin{bmatrix} K_x & K_{xz}\\ K_{zx} & K_z\end{bmatrix} \end{array}\begin{array}{c} n \\ p \end{array}, \quad P = P^T = \begin{array}{c} \begin{array}{cc} n & p \end{array} \\ \begin{bmatrix} P_x & P_{xz}\\ P_{zx} & P_z\end{bmatrix} \end{array}\begin{array}{c} n \\ p \end{array}, \quad (37)$$

it is clear from equations (14), (37), and the lemma that

$$E\left[\frac{\partial J}{\partial \underline{z}_0}\right] = E\left[\underline{\lambda}_z(0)\right] = K_{zx}(0)\,\underline{x}_0 + K_z(0)\,\underline{z}_0 = 0\,,$$

or

$$\underline{z}_0 = -K_z^{-1}(0)\;K_{zx}(0)\;\underline{x}_0\,. \quad (38)$$

It will be assumed that an inverse of K_z exists. Simultaneous solution of equations (27), (31) through (35), and (38) yields the optimal $\hat{N}$, $\underline{z}_0$.

CALCULATION OF $\hat{N}$, G_1, AND G_2

It is clear from above that the optimal solution will involve solving a nonlinear two point boundary value problem (TPBVP). Numerous numerical methods have been developed to solve TPBVP (18), and the matter will not be discussed here except to point out two straightforward algorithms for which there is no assurance of convergence.

Method 1:

a. Specify an $\hat{N}$, G_1, and G_2, which if possible stabilizes the system (usually not a small task);

b. Integrate equation (27) to obtain K(t) on $[0, t_f]$;

c. Integrate equation (31) using equation (38) for boundary information to obtain P(t) on $[0, t_f]$;

d. Perform integrations required in equation (33);

e. Solve equation (33) for a new $\hat{N}$;

f. Return to steps b and c to update K and P;

g. Revise G_1 and G_2 according to equations (34) and (35) and repeat the entire process until the convergence is achieved.

Method 2:

a. Proceed as in equation (3) except that in updating $\hat{N}$, K, and P use equation (31) to obtain K;

b. Solve equations (27) and (33) simultaneously for N and P.

DISCUSSION AND CONCLUSION

Two special cases of equations (31) through (35) are of interest. In the first case, when K_1 and K_2 are infinite, thereby permitting no parameter variation, matrices G_1 and G_2 go to zero. For output feedback (P = 0) only, equations (27), (31), and (33) reduce respectively to

$$\dot{K} + (A+BNC)^T K + K(A+BNC) + Q + C^T N^T RNC = 0 \quad K(t_f) = 0,$$

$$\dot{P} = (A+BNC)P + P(A+BNC) + DWD^T, \quad P(0) = M_0,$$

$$RNC\left[\int_0^{t_f} P(t)\,dt\right] C^T + B^T\left[\int_0^{t_f} K(t)P(t)\,dt\right] C^T = 0. \tag{39}$$

These equations are comparable to those obtained by McLane (4). McLane, however, assumes that $N = N(t)$ which, as a result, yields a modified form of equation (39).

The second special case occurs when the system is noise free; that is, $S = 0$ and $\hat{\nu} = 0$. Equation (31) becomes homogeneous and equation (33) assumes the same form as equation (39) with matrices R, N, C, and B "hatted." As $t_f \to \infty$, it can be shown that in absence of noise and parameter uncertainty,

$$M \triangleq \int_0^\infty P(t)\, dt$$

satisfies

$$(\hat{A} + \hat{B}\hat{N}\hat{C})M + M(\hat{A} + \hat{B}\hat{N}\hat{C}) + M_0 = 0, \qquad (40)$$

while equations (33) and (27) reduce respectively to

$$\hat{R}\hat{N}\hat{C}\, M\hat{C}^T + \hat{B}^T KM\hat{C}^T = 0, \qquad (41)$$

and

$$(\hat{A} + \hat{B}\hat{N}\hat{C})^T K + K(\hat{A} + \hat{B}\hat{N}\hat{C}) + Q + \hat{C}^T\hat{N}^T\hat{R}\hat{N}\hat{C} = 0. \qquad (42)$$

The conditions of equations (40), (41), and (42), except for variations due to a difference in criteria, are basically those discussed in reference (8). The design of a constant constrained optimal controller for linear systems subjected to stationary random inputs, state-dependent, control-dependent, and measurement noise, has been described.

It has been shown that the optimal controller gains may be determined by solving a nonlinear matrix TPBVP. The extent to which this approach will be applied in practice depends very much on the case with which computational problems can be solved. This is true of other approaches presented in references (2), (3), (4) and (9), (10), (11). The present approach is an alternative formulation of the problem of controlling a system with state-dependent and control-dependent noise. It is also a generalization of the problem of references (3) and (4), in the sense that no assumptions regarding intensities of state-dependent and control-dependent noise or of the availability of complete state feedback have been made, and that certain dynamics in the controller have been added to generate an estimate of feedback using only available outputs.

It should be pointed out that certain questions regarding existence and uniqueness of the solutions certainly exist (7). In addition, it has been assumed that the controller of dimension p is sufficient to stabilize the system. Uniqueness of the optimal compensator is not expected. For example, if y is scalar, G is arbitrary; then H can adjust for any value of G.

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